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Exam TWO, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let $T : V \to V$ be a linear transformation that is invertible, where V is an inner product vector space over R. Assume that $T^* = T^{-1}$. Convince me that $< T(v), T(w) >=< v, w >$ for every $v, w \in V$.

Proof. Let $v \in V$. Then $< T(v), T(w) >=< v, T^*T(w) >=< v, T^{-1}T(v) >=< v, v >$

QUESTION 2. Let T be a linear transformation from a vector space V over R to R such that $T(v_1) = 2$, $T(v_2) = 4$, and $T(v_3) = 7$, where $B = \{v_1, v_2, v_3\}$ is a basis of V. Convince me that there is a UNIQUE point $Q \in R^3$ such that $T(v) =< Q, X >$, where $X = [v]_B$ (the coordinate of v with respect to B), and $< \ , \ >$ is the normal dot product on R^3.

Proof. Let $v \in V$. Then $v = av_1 + bv_2 + cv_3$. Hence $[v]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Now $M_B = [2 \ \ 4 \ \ 7]$ is the matrix presentation of T with respect to B. Hence $T(v) = M_B[v]_B$. Thus let $Q = (2, 4, 7) \in R^3$. Then $T(v) =< Q, [v]_B >$. Now we show that Q is unique. Assume $F = (m, n, d) \in R^3$ such that $T(v) =< F, [v]_B >$. Then $T(v_1) =< F, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} >= m = 2$, $T(v_2) =< F, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} >= n = 4$, and $T(v_3) =< F, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} >= d = 7$. Thus $F = Q$.

QUESTION 3. Let $T : P_5 \to R^4$ such that $M_{B,B'} = \begin{bmatrix} 1 & 2 & 4 & 6 & -2 \\ 0 & 2 & 4 & 3 & 5 \\ 0 & 4 & 8 & 6 & 10 \\ 3 & 6 & 12 & 18 & -6 \end{bmatrix}$ be the matrix presentation of T with respect to $B = \{x^4, 1 + x^4, 1 + x + x^4, x^3 + x^4, x^2 + x^4\}$ and $B' = \{(2, 4, 6, 6), (-2, 4, 6, 6), (-2, -4, 6, 6), (-2, -4, -6, 6)\}$.

(i) Find the fake standard matrix presentation of T.

To find M_f (fake M), we use $\{1, x, x^2, x^3, x^4\}$ as the standard basis of P_5 and $\{e_1, e_2, e_3, e_4\}$ as the standard basis of R^4. Let $Q = \begin{bmatrix} 2 & -2 & -2 & -2 \\ 4 & 4 & -4 & -4 \\ 6 & 6 & 6 & -6 \\ 6 & 6 & 6 & 6 \end{bmatrix}$. Note that x^4 is viewed as $(0, 0, 0, 0, 1)$ in R^5 (since I am using $\{1, x, x^2, x^3, x^4\}$ as the standard basis of P_5, if you use $\{x^4, x^3, x^2, x, 1\}$ as the standard basis of P_5, then x^4 is viewed as $(1, 0, 0, 0, 0)$ in R^5. So let $P = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Hence we know that $M_{B,B'} = Q^{-1}M_fP$. Thus $M_f = QM_{B,B'}P^{-1}$. Now use the available (multiplication, Inverse) software and do the calculation (make sure that you know how to use the software correctly).

(ii) Use (i) and find $Range(T)$ and $Z(T)$.

To find Range(T): Put M_f in the available software, Transform M_f to echelon form, say B. Stare at the columns in B that have the leaders. Here, TWO columns in B will have the leaders. So $IN(Range(T)) = 2$. YOU MUST FIND THE CORRESPONDING TWO COLUMNS in M_f (class notes). Thus $Range(T) = span\{$ The corresponding two columns in $M_f\}$.

To find Z(T): Solve the homogeneous system $M_fX = 0$. Put the system in the available software. The software will not write it as span. From class notes, you know how to write it as span. In this question, the solution set of the homogeneous system = $span\{$3 independent points in $R^5\}$. Note that $Z(F)$ "lives" inside P_5. So translate each point to a polynomial in P_5 (see class notes). Thus $Z(T) = span\{P_1, P_2, P_3\}$.

(iii) Find $T(4x^2+x^4)$. Then find all (describe all) elements in P_5, say v, so that $T(v) = T(4x^2+x^4)$.

To find $T(4x^2+x^4)$. Do this multiplication (using the software) $M_f \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}$. Done.

This is an application of a question in one of the home works. $T^{-1}(4x^2+x^4) = \{4x^2+x^4+h \mid h \in Z(T)\}$. You already calculated $Z(T)$. Done.

QUESTION 4. Given $B = \{T_1, T_2, T_3, T_4\}$ is a basis for $Hom(P_2, P_2)$, where $T_1 : P_2 \to P_2$ such that $T_1(a_1+a_2x) = (a_1+a_2)+a_1x$ and $T_2 : P_2 \to P_2$ such that $T_2(a_1+a_2x) = (a_1+a_2)x$. Find T_3 and T_4. (i.e., you must show that T_1, T_2, T_3, T_4 are independent)

All of you got it right. For example let $T_3(a_1+a_2x) = a_2$, $T_4(a_1+a_2x) = a_2x$

QUESTION 5. Let V be an inner product space over R. Convince me that $||v+w||^2 = ||v||^2+||w||^2$ for every orthogonal elements $v, w \in V$.

$||v+w||^2 =< v+w, v+w >=< v,v > +2 < v,w > + < w,w >= ||v||^2 + ||w||^2$ (since v, w are orthogonal, i.e., $< v,w >= 0$.)

QUESTION 6. Let $W = span\{A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$ Find a basis for $W^{\perp}$ (note $< A,B >= Trace(B^T A)$)

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Hence $Trace(B^T A) = 0$ and $Trace(B^T K) = 0$. Hence $a+b = 0$ and $a+d = 0$. Solution set to the homogeneous system is $\{(a, -a, c, -a) \mid a, c \in R\}$ = span$\{(1,-1,0,-1), (0,0,1,0)\}$. Now translate to matrices. Hence $W^{\perp} = span\{\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$. **[some of you used the fake-dot product on $R^{2\times 2}$, so I accepted that.. but next time I will not]**

QUESTION 7. Let $T : R^4 \to R^4$ be a linear transformation (operator) such that the matrix presentation of T with respect to the basis $B = \{(1,1,1,1), (-1,1,1,1), (-1,-1,1,1), (-1,-1,-1,1)\}$ is $M_B = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(i) Find $C_T(x)$ and $m_T(x)$. By staring, M_B is the companion matrix of the polynomial $x^4 - 5x^2 + 4$. Hence we know (by class notes) that $C_T(x) = m_T(x) = x^4 - 5x^2 + 4$.

(ii) Convince me that T is diagnolizable. Since $m_T(x) = x^4 - 5x^2 + 4 = (x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$ (i.e., $m_T(x)$ is a product of distinct linear factors), by class notes T is diagnolizable.

(iii) Find the standard matrix presentation of T^2

Two solutions are accepted:

(1) Assume B is the basis for the domain and the co-domain. Hence $P = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

We know that $M_B = P^{-1}MP$. Hence $M = PM_BP^{-1}$ is the standard matrix presentation of T. By class notes (old HW), the standard matrix presentation of T^2 is M^2. Use the available software (multiplication, inverse) to find M and M^2.

(2)Assume B is the basis for the domain and the standard basis $\{e_1, e_2, e_3, e_4\}$ is the basis for the co-domain. Hence $P = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, and $Q = I_4$.

We know that $M_B = I_4^{-1}MP$. Hence $M = I_4M_BP^{-1} = M_BP^{-1}$ is the standard matrix presentation of T. By class notes (old HW), the standard matrix presentation of T^2 is M^2. Use the available software (multiplication, inverse) to find M and M^2.

(iv) Let $F = 5T^2 - T^4 - I$ (then F is an operator from R^4 into R^4). Convince me that 3 is an eigenvalue of F. Find an orthonormal basis of $E_3(F)$.

Process of thinking: By staring $5T^2 - T^4 - I$ is some how related to $C_T(x) = x^4 - 5x^2 + 4$ (some of you observed that). We know (class notes) $C_T(T) = T^4 - 5T^2 + 4I = 0$. Thus $3I = 5T^2 - T^4 - I = F$. Hence $3I(v) = F(v) = 3v$ for every $v \in R^4$. Hence 3 is an eigenvalue of F and $E_3(F) = R^4$. Hence an orthonormal basis is $\{e_1, e_2, e_3, e_4\}$. DONE

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com